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- entirely similar?

Permutation Transformations

The $6N$-dimensional phase space of a system of N point particles has phase points s of the form of a $6N$ tuple:

$$s = ((\mathbf{r}, \mathbf{p})_1, ..., (\mathbf{r}, \mathbf{p})_N) \tag{1}$$

where $\mathbf{r} = (r_x, r_y, r_z)$ and $\mathbf{p} = (p_x, p_y, p_z)$.

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where $\mathbf{r} = (r_x, r_y, r_z)$ and $\mathbf{p} = (p_x, p_y, p_z)$. Consider the family of transformations $\{\pi_{ij}\}$ for $1 \leq i, j \leq N$ such that, e.g.

$$\pi_{1N}(s) := ((\mathbf{r}, \mathbf{p})_N, ..., (\mathbf{r}, \mathbf{p})_1). \tag{2}$$

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- Permutation of particle labels?
- Permutation of particles among the states?
- *Permutation of states over particles.*

Simple Example

(a) $s = (1, -1, 2, 1)$.

(b) $\pi(s) = (2, 1, 1, -1)$.

Figure: Two states related by a permutation. In each case, there is one particle with $(r, p) = (1, -1)$ and another with $(2, 1)$ and yet the states are represented by two distinct points in the phase space. The black dot depicts the particle represented by the first factor position and the white dot depicts the particle represented by the second factor position.

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Regard the phases as representing distinct states of affairs:

- If you think trajectories play a role in distinguishing the particles.

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- What harm can representational redundancy do?
- When we start putting probability measures on the phase space.
- If phase space over-represents, the probability measure over-counts.
- We use probability measures to calculate expectation values of physical quantities.
- Relatedly (less obvious) this over-counting apparently leads to a non-extensive entropy function in statistical mechanics.

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- Now we count the microstates.

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Take the logarithm and apply Stirling's approximation: $\ln x! \approx x \ln x - x$ for large x. Then the entropy reads:

$$S = kN \ln N + ...$$

Because of this first term, the entropy is clearly not extensive. Divide W by $N!$? But what justification?

The Orthodox Solution: Quantum Statistics

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In the 'classical limit' $C_s \gg N_s$

$$W_{\text{FD and BE}} \approx \prod_s \frac{{C_s}^{N_s}}{N_s!} = W_B/N! \tag{5}$$

Symmetrisation Postulate

"It is not possible to understand classically why we must divide [...] by $N!$ to obtain the correct counting of states. The reason is inherently quantum mechanical. Quantum mechanically, atoms are inherently indistinguishable in the following sense: A state of the gas is described by an N-particle wave function, which is either symmetric or antisymmetric with respect to the interchange of any two particles. A permutation of the particles can at most change the wave function by sign, and it does not produce a new state of the system. Hence we should divide [...] by $N!$. This rule of counting is known as the "correct Boltzmann counting". It is something that we must append to classical mechanics in order to get right answers." (Huang 1963)

Uncertainty Relations

"So long as we are dealing with classical systems, the $N!$ microsituations are in fact different and can be distinguished from one another. In quantum mechanical systems, however, we can no longer distinguish them. This is certainly due to the fact that we cannot follow the atoms so closely along their orbits that we can, for instance, know which atom is which after a collision. This again is a consequence of Heisenberg's famous relations." (Ter Haar 1954)

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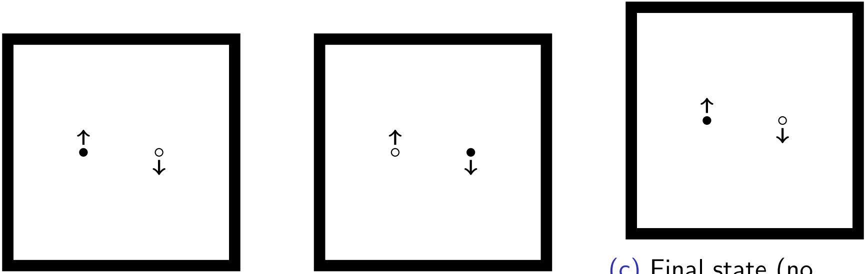
- Driving the orthodox solution is the assumption that classical particles are distinguishable and quantum particles are indistinguishable.
- Should we regard Gibbs' Paradox as some kind of Kuhnian anomaly? A problem only quantum theory can resolve?
- Or can it (should it?) be solved purely classically?
- Is there any structural feature of the problem which tells us which option to take?

Dieks' Classical Particle Distinguishability

[C]lassical particles can be named and distinguished by their different histories. A process in which two classical particles of the same kind are interchanged can therefore certainly produce a different microstate. Indeed, imagine a situation in which there is one particle at position [r_1] and one particle at position [r_2], and in which at a later instant there is again one particle at [r_1] and one at [r_2]; suppose that their respective momenta are the same as before. What has happened in the meantime? There are two possibilities: either the particle that was first at [r_1] is later again at [r_1] and the particle that was first at [r_2] is later again at [r_2], or the particles have exchanged their positions. The latter case would clearly be different from the former one: it corresponds to a different physical process. Although it is true that the two final situations cannot be distinguished on the basis of their instantaneous properties, their different histories show that the particle at [r_1] in one final situation is not the same as the particle at [r_1] in the other final situation. (Dieks 2013)

Trajectories = Distinguishability



(a) Initial state

(b) Final state (swap)

(c) Final state (no swap)

Figure: (a) is the initial state of a system consisting of two particles in a container. The arrows indicate their velocities. States (b) and (c) are the final states of particles after they follow trajectories which lead them to swapping and not swapping, respectively. The black dot depicts the particle represented by the first factor position and the white dot depicts the particle represented by the second factor position.

Evaluating Dieks' Argument

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Note that the system in which the particles swap must have a different dynamics from the system in which they don't:

$$\varphi_{\neg\text{swap}} : s \mapsto s \tag{6}$$

$$\varphi_{\text{swap}} : s \mapsto \pi_{12}(s) \tag{7}$$

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- If the particles are identical (they share all intrinsic properties like charge, mass and spin), then the states $\pi_{ij}(s)$ and s (for $1 \leq i, j \leq N$) represent one and the same physical state.

Saunders Classical Particle Indistinguishability

- Remove representational redundancy
- If the particles are identical (they share all intrinsic properties like charge, mass and spin), then the states $\pi_{ij}(s)$ and s (for $1 \leq i, j \leq N$) represent one and the same physical state.
- Remove this representational redundancy by passing to the reduced phase space, $M' = M/S_N$: the quotient of the phase space under the action of S_N, the permutation group. Points $s \in M'$ may be understood as equivalence classes $[s]$ each containing $N!$ elements all related to the $s \in M$ by the permutations $\pi \in S_N$.

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- But isn't it puzzling that we are considering Gibbs' paradox from the Boltzmann approach?
- And isn't it funny that sometimes the $N!$ is talked of as if it is a 'correction factor'? Can't it be rigorously derived?

A Simple Motivation

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Consider a system which flips a red coin (R) and a green coin (G) each of which can land Heads (H) or Tails (T). Suppose an observer of this system is colourblind. Then the set of physical states is

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So:

- We must view the space which 'carries' the probability measure in statistical mechanics as a sample space, not the phase space.

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- If we quotient with this equivalence relation, we do not end up with a state space, but rather a sample space.

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- Use this notion to construct the sample space Ω from the state space M.

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Dynamical Indistinguishability

Two particles of an N particle system, represented by factor positions i and j in the N tuple, are *dynamically indistinguishable* if and only if the Hamiltonian h of the system satisfies $h(s) = h(\pi_{ij}(s))$ for all states s.

Examples

Consider the permutation symmetries of the following Hamiltonians:

- $h_1 = (p_1^2 + p_2^2)/2m$;
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- $h_2(s) = h_2(\pi_{13}(s))$ and $h_2(s) = h_2(\pi_{24}(s))$ for all s;
- But $h_2(s) \neq h_3(\pi_{23}(s))$ and $h_2(s) \neq h_3(\pi_{14}(s))$

Gibbs' Statistical Mechanics

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This collection of systems is known as the *ensemble* and it is this that is the object of study in Gibbs' statistical mechanics:

> *And here we may set the problem, not to follow a particular system through its succession of configurations, but to determine how the whole number of systems will be distributed among the various conceivable configurations and velocities at any required time, when the distribution has been given for some one time. (Gibbs 1902)*

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- μ is a probability measure on M if $\mathcal{N} = n(M)$.

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- It can happen that $\mathcal{N} < n(M)$ when distinct points in M are observationally indistinguishable and hence correspond to the very same point in Ω.
- If this happens then $\mu(M) > 1$ and $\mu(\Omega) = 1$ for $\Omega \subset M$.

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- If this happens then $\mu(M) > 1$ and $\mu(\Omega) = 1$ for $\Omega \subset M$.
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- $\mathcal{P} \equiv \mu$ if $\mathcal{N} = n(M)$.

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- So $\mathcal{P}(R) = \mu(R)/N!$.